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Energy Absorption in Cylinders Containing a Uniformly Distributed Source[1]

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As discussed in our previous paper (1), calculation of energy absorption in tissue containing a gamma-ray emitting radioisotope requires knowledge of the absorption fraction (ratio of absorbed energy to emitted energy), as well as the energetics of the isotope (2). In that paper the absorption fraction was calculated for cylinders containing an axial source. The present paper presents the solution for cylinders containing a uniformly distributed source. Based on this solution, the absorption fraction is tabulated for a range of cylinder sizes and shapes encompasing those most likely to be of medical interest. Only limited solutions of this problem have previously been available (3, 4, 5, 6).

The data we have obtained provide improved estimates of energy absorption for medical as well as non-medical dosimetry. Derivations are given in the Appendix; results are given in Figure 1 and Table 1.

Absorption fractions were calculated on a Philco 212 Computer for a wide range of cylinder shapes from rods to discs. Likewise, the range of μR is sufficient to encompass absorption fractions from 0.5 to 65 per cent for flat discs, diameter ten times height. This shape is probably flatter than any that will be required for medical dosimetry. At the other extreme, data for the infinite cylinder computed by Case *et al.* (4) have been used to extend those obtained by our computer program.

A computation comparable to that presented here is the recently published table of Focht *et al* (6), which appeared after this work was largely completed. These authors present their data in terms of the geometric factor and for $\mu = 0.028\ \text{cm}^{-1}$. Using the relation

$$\Phi = \frac{\mu \bar{g}}{4\pi}$$

these data can be converted to absorption fraction and generally prove similar to those given here. For the lowest values of g, our results are as much as 15

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per cent higher. The source of this discrepancy is not apparent since the analysis for their computation has not been given in detail.

The assumption of exponential absorption implies that the data will be strictly correct only when photoelectric and single Compton interactions predominate. The appropriate absorption coefficient to use is $\frac{\mu_{en}}{\rho}$ (2). When an appreciable amount of multiply scattered radiation is absorbed, the values obtained using $\frac{\mu_{en}}{\rho}$ will be slightly low. The amount of energy absorbed in multiple interactions depends on the size and shape of the absorber as well as the photon energy. An estimate of the error involved can be obtained by comparing the analytical values of the absorption fraction for spheres (3) with the values

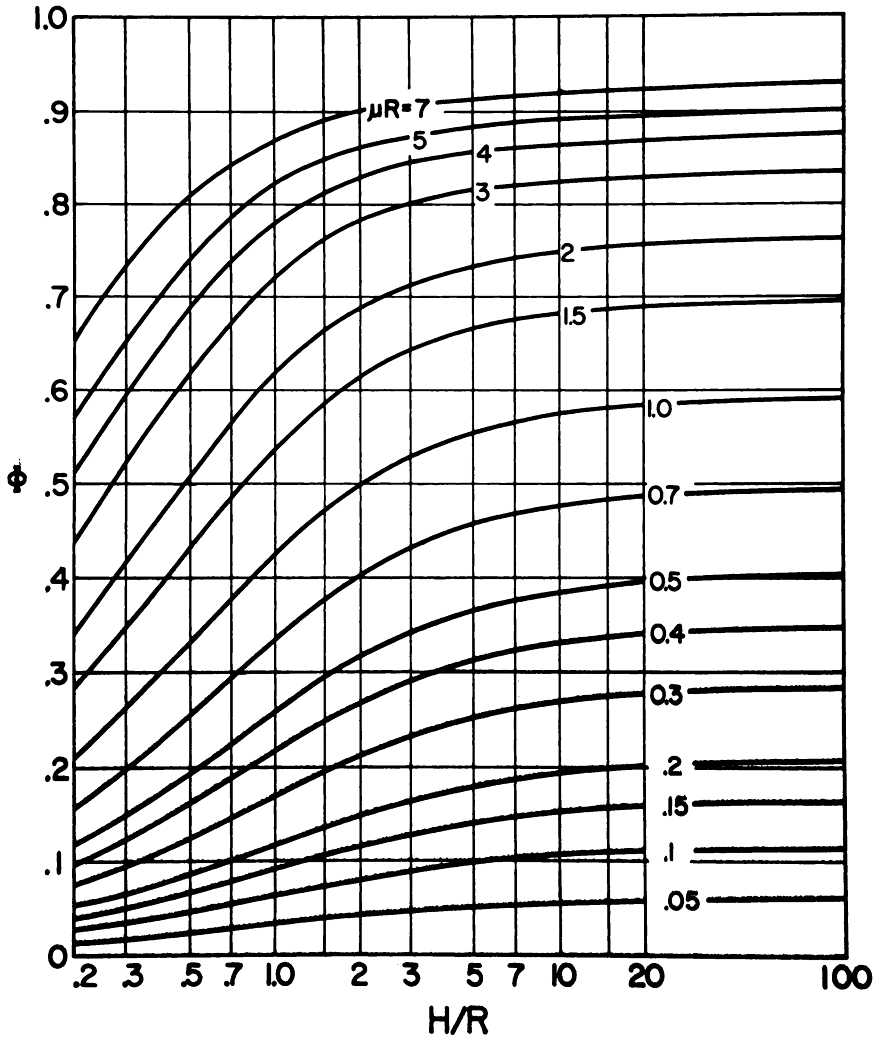

Fig. 1. Absorption fraction as a function of μR and H/R (μ = absorption coefficient, R = cylinder radius, H = cylinder height).

obtained by Ellett *et al.* (7). In this comparison, the error for photon energies greater than 0.2 MeV is negligible. The error for lower energies depends slightly on phantom size and more strongly on the photon energy; analytical absorption fractions less than about 0.5 are roughly 20 per cent low for photon energies of 0.05 to 0.15 MeV and 10 percent low for photon energies of 0.15 to 0.2 MeV.

SUMMARY

Analysis and partial integration of the absorption fraction for cylinders containing a uniformly distributed radioactive isotope are presented. The absorption

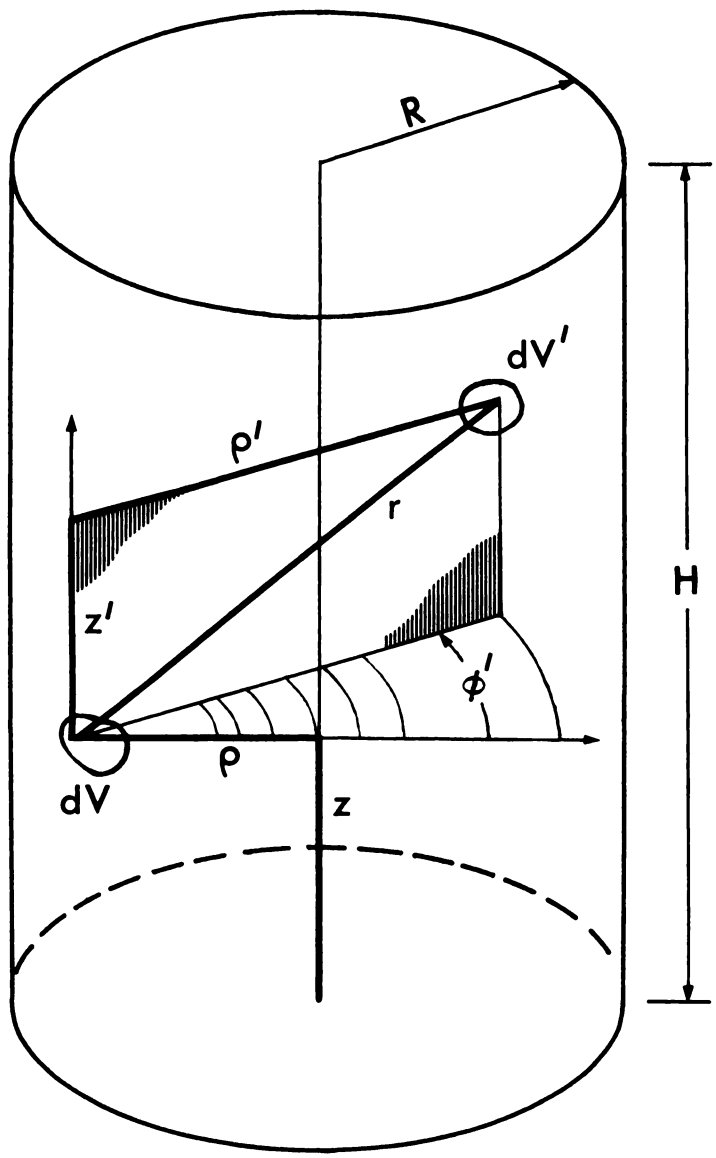

Fig. 2.

TABLE I: ABSORPTION FRACTION

μR \ H/R	*0.2*	*0.3*	*0.5*	*0.7*	*1.0*	*1.5*	*2*	*3*	*5*	*7*	*10*	*15*	*20*	*100*	∞
7.0	.652	.730	.804	.840	.866	.887	.897								
5.0	.568	.651	.739	.783	.818	.846	.860	.872							.901
4.0	.511	.595	.689	.738	.779	.810	.827	.843	.854						.877
3.0	.438	.520	.616	.671	.718	.757	.777	.797	.813						.837
2.0	.342	.415	.508	.564	.617	.663	.688	.712	.733	.743	.747				.764
1.5	.281	.346	.431	.485	.537	.585	.613	.641	.665	.674	.680				.698
1.0	.208	.259	.330	.377	.425	.471	.500	.530	.554	.565	.575	.580	.582		.593
0.7	.156	.196	.253	.293	.333	.375	.401	.430	.455	.467	.474	.482	.486		.494
0.5	.117	.148	.193	.225	.258	.294	.317	.342	.366	.376	.385	.392	.396		.404
0.4	.096	.122	.160	.187	.216	.247	.267	.290	.313	.323	.331	.336	.340		.348
0.3	.074	.094	.124	.146	.169	.194	.211	.231	.250	.260	.267	.273	.275		.284
0.2	.051	.065	.086	.101	.118	.137	.149	.164	.180	.186	.193	.198	.201	.207	.207
0.15	.038	.049	.066	.077	.090	.105	.115	.127	.140	.146	.150	.155	.158	.162	.163
0.10	.026	.033	.045	.053	.062	.072	.079	.088	.096	.101	.105	.108	.110	.115	.115
0.07	.018	.024	.032	.037	.044	.051	.056	.063	.069	.073	.076	.078	.079	.082	.083
0.05	.013	.017	.023	.027	.032	.037	.041	.045	.050	.053	.055	.057	.058	.061	.061
0.04	.011	.014	.018	.022	.025	.030	.033	.037	.041	.043	.044	.046	.047	.050	.050
0.03	.008	.010	.014	.016	.019	.022	.025	.028	.031	.032	.034	.035	.036	.038	.038
0.02	.005	.007	.009	.011	.013	.015	.017	.019	.021	.022	.023	.023	.024	.026	.026

fractions for a range of cylinder sizes, shapes and absorption coefficients are tabulated.

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APPENDIX

A cylinder of radius R and height H contains a uniformly distributed γ-ray source of intensity σ energy emitted per unit volume. Let

$U =$ energy absorbed,

$\Phi =$ absorption fraction (ratio of absorbed energy to emitted energy),

$\mu =$ linear absorption coefficient, and

$dU =$ energy absorbed in volume dV from energy emitted by source dV'

$$= \frac{\mu\sigma}{4\pi}\frac{e^{-\mu r}}{r^2}\, dV\, dV'.$$

Then $U = \dfrac{\mu\sigma}{4\pi}\displaystyle\int dV \int \frac{e^{-\mu r}}{r^2}\, dV'$ and

$$\Phi = \frac{\mu}{4\pi^2 R^2 H}\int dV \int \frac{e^{-\mu r}}{r^2}\, dV'.$$

To integrate the dV cylindrical coordinates are chosen coaxial with the cylinder with the origin at the base. To integrate the dV′ the origin is located at dV with the coordinate axis parallel to the cylinder axis (Fig. 2). Then

$$\Phi = \frac{\mu}{4\pi^2 R^2 H}\int_0^R \rho d\rho \int_0^{2\pi} d\phi \int_0^H dz \int_0^{2\pi} d\phi' \int_{-z}^{H-z} dz' \int_0^{\rho\cos\phi' + \sqrt{R^2 - \rho^2 \sin^2\phi'}} \frac{e^{-\mu(\rho'^2 + z'^2)^{\frac{1}{2}}}}{\rho'^2 + z'^2}\, \rho' d\rho'.$$

The integral over ϕ is 2π. Interchanging the order of integration, and because the integrand is an even function of z′ and does not depend on z

$$\Phi = \frac{\mu}{\pi R^2 H}\int_0^R \rho d\rho \int_0^{2\pi} d\phi' \int_0^H dz \int_0^z dz' \int_0^{\rho\cos\phi' + \sqrt{R^2 - \rho^2 \sin^2\phi'}} \frac{e^{-\mu(\rho'^2 + z'^2)^{\frac{1}{2}}}}{\rho'^2 + z'^2}\, \rho' d\rho'.$$

In terms of the dimensionless variables

$E = \frac{H}{R}$, $k = \mu R$, $t = \frac{z'}{R}$, $y = \frac{z}{R}$, $s = \frac{\rho}{R}$, $x = \frac{\rho'}{R}$ and $\alpha = \phi'$

$$\Phi = \frac{k}{\pi E}\int_o^1 sds\int_o^{2\pi} d\alpha\int_o^E dy\int_o^y dt\int_o^{X_0}\frac{e^{-k(x^2+t^2)^{\frac{1}{2}}}}{x^2+t^2}\,xdx$$

with $X_0 = s\,cos\alpha + \sqrt{1 - s^2\,sin^2\alpha}.$

Expanding the exponential and integrating over x gives

$$\Phi = \frac{k}{\pi E}\int_o^1 sds\int_o^{2\pi} d\alpha\int_o^E dy\int_o^y \frac{1}{2}\,log\,(X_0^2 + t^2)\,dt \qquad (1)$$

$$+\frac{k}{\pi E}\int_o^1 sds\int_o^{2\pi} d\alpha\int_o^E dy\int_o^y \sum_{n=1}^{\infty}\frac{(-k)^n}{n\cdot n!}(X_0^2+t^2)^{\frac{n}{2}}dt \qquad (2)$$

$$-\frac{k}{\pi E}\int_o^1 sds\int_o^{2\pi} d\alpha\int_o^E dy\int_o^y \left[logt + \sum_{n=1}^{\infty}\frac{(-k)^n}{n\cdot n!}t^n\right]dt. \qquad (3)$$

Using standard tables (8) eq. (1) can be partially integrated to

$$(1) = -\frac{3kE}{4} + \frac{k}{\pi E}\int_o^1 sds\int_o^{2\pi} d\alpha\left[\frac{(E^2 - X_0^2)}{4}\,log\,(X_0^2 + E^2)\right.$$

$$\left. + EX_0\,tan^{-1}\frac{E}{X_0} + \frac{X_0^2}{2}\,log\,X_0\right].$$

Eq. (3) can be integrated to give

$$(3) = -\frac{kE}{2}\,log\,E + \frac{3kE}{4} + \sum_{n=1}^{\infty}\frac{(-kE)^{n+1}}{n(n+2)!}.$$

To reduce (2), expand binomially for n even. For n odd, use the formula

$$\int (x^2 + a^2)^{\frac{n}{2}}\,dx$$

$$= \frac{n!}{2\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!}\sum_{j=0}^{\frac{n-1}{2}}\frac{(j!)^2}{(2j+1)!}\left(\frac{a}{2}\right)^{n-2j-1}x(x^2+a^2)^{j+\frac{1}{2}}$$

$$+\frac{2n!}{\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!}\left(\frac{a}{2}\right)^{n+1}log\,(x + \sqrt{x^2+a^2}).$$

Then,

$$2) = \frac{1}{4\pi}\sum_{n=2,4,6\ldots}^{\infty}\frac{k^{n+1}\left(\frac{n}{2}-1\right)!}{n!}\sum_{j=0}^{\frac{n}{2}}\frac{E^{2j+1}}{(2j+1)\,(j+1)!\left(\frac{n}{2}-j\right)!}\int_0^1 sds\int_0^{2\pi}X_0^{n-2j}\,d\alpha$$

$$-\frac{1}{\pi E}\sum_{n=1,3,5\ldots}^{\infty}\frac{k^{n+1}}{n\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!}\sum_{j=0}^{\frac{n-1}{2}}\frac{(j+1)!j!}{2^{n-2j-1}(2j+3)!}\int_0^1 sds\int_0^{2\pi}\left[X_0^{n-2j-1}(X_0^2+E^2)^{j+\frac{3}{2}}-X_0^{n+2}\right]d\alpha$$

$$-\frac{1}{\pi E}\sum_{n=1,3,5}^{\infty}\frac{k^{n+1}}{n2^n\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!}\int_0^1 sds\int_0^{2\pi}X_0^{n+1}\left[E\log\left(E+\sqrt{X_0^2+E^2}\right)\right.$$

$$\left.+X_0-\sqrt{X_0^2+E^2}-E\log X_0\right]d\alpha\,.$$

The expression for n even can be integrated by noting that

$$\int_0^1 sds\int_0^{2\pi}X_0^{n-2j}\,d\alpha=\frac{\pi}{2(n-2j+1)}\binom{n-2j+2}{\frac{n}{2}-j+1}\,.$$

Using this relation and rewriting the summation indices over n gives

$$\Phi=\frac{k}{\pi E}\int_0^1 sds\int_0^{2\pi}\left[\frac{(E^2-X_0^2)}{4}\log(X_0^2+E^2)+EX_0\tan^{-1}\frac{E}{X_0}+\frac{X_0^2}{2}\log X_0\right]d\alpha$$

$$-\frac{kE}{2}\log E+\sum_{n=1}^{\infty}\frac{(-kE)^{n+1}}{n(n+2)!}$$

$$+\frac{1}{4}\sum_{n=1}^{\infty}\frac{k^{2n+1}(n-1)!}{(2n)!}\sum_{j=0}^{n}\frac{(2j)!\,E^{2n-2j+1}}{(2n-2j+1)\,(n-j+1)!\,(j+1)!(j!)^2}$$

$$-\frac{1}{\pi E}\sum_{n=0}^{\infty}\frac{k^{2n+2}}{(2n+1)\,(n+1)!\,n!}\sum_{j=0}^{n}\frac{(j+1)!\,j!}{4^{n-j}(2j+3)!}\int_0^1 sds\int_0^{2\pi}\left[X_0^{2n-2j}(X_0^2+E^2)^{j+\frac{3}{2}}-X_0^{2n+3}\right]d\alpha$$

$$-\frac{1}{2\pi E}\sum_{n=0}^{\infty}\frac{k^{2n+2}}{4^n(2n+1)\,(n+1)!\,n!}\int_0^1 sds\int_0^{2\pi}X_0^{2n+2}\left[E\log\left(E+\sqrt{X_0^2+E^2}\right)\right.$$

$$\left.+X_0-\sqrt{X_0^2+E^2}-E\log X_0\right]d\alpha \qquad (4)$$

Some of the above integrals can be further reduced, but only at the expense of increasing the complexity of the computer program. An equation equivalent to (4) was programmed and run to obtain the values appearing in Table 1 and Figure 1. Since the integrals and sums over the index j depend only on the parameter E, their completion for a given E and k permits computation for lower values of k without further integration. The data are expected to be accurate to within 1 or 2 parts in 10^3. The program can be used to obtain more accurate data or data for other values of k or E than those presented.

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