Relationship of Surface Area on Roentgenograms and Radioisotopic Scans to Organ Volumes

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From roentgenograms and scans obtained by use of radioisotopes, it is possible to determine uncorrected organ dimensions and apparent cross sectional areas. The relationship of such measured values to the volume (and weight, if the density is known) of the organ is of interest. If reproductible scanning techniques and reliable predicting equations were available, they might permit an assessment of organ weights and of growth during life. In the present communication, we examine possible simple theoretical relationships between the measured dimensions or cross sectional area and the organ's volume and illustrate the discussion by examples taken from the literature on the normal thyroid, spleen, and kidney.

ANALYSIS

To examine the problem in terms of dimensions, we find that the cross section area of the organ appears as an apparent area or surface on the radioisotopic scan. The area of this surface (S) (or product of length and width) found on roentgenograms and scans, has the dimensions of distance squared. Since a volume (V) has the dimensions of distance cubed, the simplest theoretical relationship would be of the type:

\[ V = k \cdot S^{3/2} \]  

(1)

For simplicity in handling the data, this equation can be recast into linear form by taking logarithms:

\[ \log V = \log k + \frac{3}{2} \log S \]  

(II)

Since k is a constant, the equation can be rewritten as:

\[ \log V = q + 1.5 \log S \]  

(III)

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The prediction is that a plot of the logarithm of the organ volume (or weight, if the density is constant) should be a linear function of the logarithm of the scan “surface area”, or apparent cross sectional area. This should be true if there remains a constant (albeit complex and usually unknown) power relationship between the cross sectional area and the organ’s dimensions.

It can be recognized that for several geometric figures the cross sectional area can be used to directly calculate the volume. Two examples are presented here.

1. For a rectangle.

\[ V = l \cdot w \cdot d \] (IV)

where \( l \) is length, \( w \) is width, and \( d \) is depth. If the depth and length are related to the width by a constant (such as \( d = a \cdot w \), and \( l = q \cdot w \)) then:

\[ V = a \cdot q \cdot w^3 \] (V)

The cross sectional area seen on a scan is:

\[ A = lw \] (V1)

or:

\[ A = qw^2 \] (VII)

Hence, we can relate the volume to the cross sectional area of this special figure:

\[ V = A^{3/2} \cdot a/q^3 \] (VIII)

or:

\[ \log V = 3/2 \log A + k \] (IX)

For the rectangle in which all sides are equal (a cube) \( a = q = 1 \), and

\[ \log V = 3/2 \log A \] (X)

2. For a sphere:

\[ V = 4/3 \cdot R^3 \] (XI)

where \( R \) is the radius. The cross sectional area is a circle:

\[ A = \pi R^2 \] (XII)

Thus, the volume of the sphere is related to the cross sectional area by:

\[ V = A^{3/2} \cdot (4/3)^{1/2} \] (XIII)

or:

\[ \log V = 3/2 \log A + f \] (XIV)

We will now compare literature data on organ apparent cross sectional areas with their reported weights by use of equation (III). That is, a logarithmic comparison of the organ’s weight will be made with the apparent cross sectional area.
Spleen

For the spleen, we can employ data reported by Holzbach and coworkers (1). These investigators tagged a sample of the patient’s blood with $^{51}$Cr and then denatured the erythrocytes by heating. Following intravenous injection, the denatured cells accumulated in the spleen permitting a lateral scan to be obtained. Holzbach et al, recorded the surface area of the lateral scan as well as the organ’s weight, but did not derive a relationship between these values. Using the table presented by Holzbach and coworkers (1) we have plotted the logarithm of the spleen weight as a function of the logarithm of the surface area of the lateral scan (Fig. 1). These values, presented in Figure 1, are described by the equation (determined by the method of least squares):

$$\log W = 1.50 \log A - 0.59$$  \hspace{1cm} (XV)

or:

$$W = 0.257 A^{1.50}$$  \hspace{1cm} (XVI)

Fig. 1. A plot of the logarithm of spleen weight as a function of the logarithm of the area determined by the splenic scanning [data from Holzbach et al (1)]. Extending the line to the 4 abnormally enlarged or contracted spleens described by these authors does not significantly change the line.
It can be noted that the exponent (1.50) does not differ appreciably from the expected value of 1.5.

Whitley and coworkers (2) measured the length and width of the spleen on roentgenograms and presented a linear formula for calculating splenic weight from the product of the length and width. Logarithmic transformations of the data were not tried by Whitley et al. (2). If we assume that the correct relationship is of the form shown by equation I, the result is:

\[ W = 0.7 \ P^{3/2} \]  \hspace{1cm} (XVII)

or:

\[ \log W = \log 0.7 + 1.5 \log P \]  \hspace{1cm} (XVIII)

where \( P \) is the product of length and width. A comparison of this equation with the linear formula of Whitley et al (2) is made in Table I. It can be observed that there is little to choose between the two equations in the region between \( P \) values of 30 and 100 (the range in which most spleen sizes occur). For larger spleen sizes, however, the linear formula of Whitley et al. (2) may be less reliable. For example, the upper two points of Whitley’s Figure 2 reveal a mean splenic weight of 1350 gm and a \( P \) of 144 cm (2). By use of the exponential equation VI, the calculated value is 1210 gm, while Whitley’s formula gives a value of but 1,113 gm.

**Thyroid**

Following administration of \(^{131}I\), the radioiodide concentrates in the thyroid can be used to obtain scans of the gland. Himanka and Larsson (3) noted that the “frontal silhouette” (or cross sectional area in our terminology) of the gland obtained this way would be related to its volume by:

\[ V = k_1 \ (\sqrt{A})^8 \]  \hspace{1cm} (XIX)

**Table I**

**A Comparison of Estimated Spleen Weights as Determined by Equation XVII and by the Regression Formula of Whitley et al. (2)**

<table>
<thead>
<tr>
<th>Product of length ( \times ) width (cm(^2))</th>
<th>Calculated from ( W = 0.7 \ P^{3/2} )</th>
<th>Calculated from ( G = 7.73 \ P - 122 )</th>
<th>Difference ( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>115</td>
<td>110</td>
<td>+4.6%</td>
</tr>
<tr>
<td>40</td>
<td>177</td>
<td>187</td>
<td>-5.3%</td>
</tr>
<tr>
<td>80</td>
<td>501</td>
<td>496</td>
<td>+1.0%</td>
</tr>
<tr>
<td>100</td>
<td>700</td>
<td>651</td>
<td>+7.6%</td>
</tr>
</tbody>
</table>

\(^{1}\)Regression formula of Whitley and coworkers.
\( \Delta \) Log formula — linear formula

\[ \times 100 \]  \hspace{1cm} Linear formula
However, these authors proceeded to plot \( V \) versus \( \log (\sqrt{A}) \) (3) which is a calculated quantity. The simpler approach is to note that:

\[
\log V = \log k_1 + 3/2 \log A \tag{XX}
\]

and to plot \( V \) versus \( A \) directly on log-log paper. Taking the points from Diagram 1 of Himanka and Larsson, their values can be replotted in log-log form. The constant \( k_1 \) was given as \( 0.33 \pm 0.06 \). That is:

\[
\log V = \log 0.33 + 1.5 \log A \tag{XXI}
\]

The average error observed by Himanka and Larsson was \( \pm 20\% \). Again the exponent (1.5) does not differ appreciably from that predicted by equation 1. It must be recognized that “normal” thyroid glands are under discussion. Those with quite aberrant shapes, nodules, or other disorders may deviate from such predicting equations (4). An important part of future studies will consist of determining the anatomic shapes for which the equations are valid.

**Kidney**

Of all the body organs, the kidneys perhaps best fit the description of a regular geometric figure. Moëll (5) pointed out that if the postmortem weight of the adult human kidneys (gm) was plotted on a logarithmic scale against the logarithm of the roentgenographic area (cm\(^2\)), a straight line was obtained (his Figure 8). The line was:

\[
\log W = 1.21 \log A - 0.18 \tag{XXII}
\]

or:

\[
W = 0.66 A^{1.21} \tag{XXIII}
\]

A fairly similar line can be observed in data presented by Ludin (6) (his Figure 10) from corrected tomograms of the human kidneys. The exponent from Ludin's figure is 1.20 which compares well with the value of 1.21 in equation XI.

**DISCUSSION**

It is also conceivable that exponents other than 3/2 might be encountered (see: for example, equations 22 and 23). Consider the case of a rectangle growing so that the length is given by some constant relationship to the width (such as \( l = q \cdot w \)) and the depth is described by a power relationship to the width, such as:

\[
d = n \cdot w^h \tag{XXIV}
\]

Hence, the cross sectional area is \( A = qw^2 \) as in equation VII and the volume is:

\[
V = q \cdot n \cdot w^{(2+h)} \tag{XXV}
\]
Thus we can write:

\[ V = A \left( \frac{2 + h}{2} \right) \cdot \left( \frac{n}{q^{h/2}} \right) \]  

(XXVII)

or:

\[ \log V = (\log(2 + h/2) \log A + \log \left( \frac{n}{q^{h/2}} \right) \]  

(XXVII)

When \( h = 1 \), equation (XXVII) reduces to equation (III). However, when \( h < 1 \), the exponent will be less than 3/2. It is because of this that logarithmic plotting of \( V \) against \( A \) is desirable rather than plotting \( V \) against \( A^{3/2} \) (since there is no assurance that the exponent will indeed be 3/2).

The discussion has been limited to organs of normal size. Those that are smaller than normal (as might be due to infarction or compromise of the blood supply), or larger than expected from tumor involvement or other process, might vary from the expected growth pattern. Hence, initial studies should carefully separate normal organs from those affected by disease states. What also remains to be carefully determined in each case is the error involved in the measurement of the cross sectional area and its effect on the estimation of the organ volume. Considering the present limited means of estimating the volume or weight of internal organs, an error of ± 15% might be acceptable as a first approach.

The technique proposed here, of plotting the organ weight as a function of the 3/2 power of the cross sectional area, was utilized because it appeared to have a theoretical foundation and gave results that were consistent with literature data. Other approaches are of course possible. Goodwin and coworkers (7) for example, proposed methods such as plotting the volume against the cube of a linear dimension. Such techniques also deserve careful evaluation in the case of each organ.

An advantage to the procedure of estimating an organ’s volume from a single scan is that but one view is required; patient time and the task of calculation is thus minimized. Such an approach might however mis-estimate the volume of an organ whose growth was abnormal in one direction. In such instances, data from more than one view might be highly desirable. For example, we have pointed out (8) that the area on both anteroposterior (\( A_n \)) and lateral (\( A_l \)) scans may be utilized. In this case, the volume would be dimensionally related to the areas by:

\[ V = (A_n \cdot A_l)^{3/4} \]  

(XXVIII)

These dimensional analyses appear to conform to the statement by Wright (9) that a "...correct hypothesis for the solution of a problem often turns out to be the least complicated one that can be thought of at the time."

SUMMARY

The relationship of the apparent organ cross-sectional area, seen on roentgenograms and on radioisotopic scans, to the volume of the organ, was analyzed in terms of the dimensions involved. A logarithmic plot of the organ’s volume (or weight, assuming a constant density) as a function of the logarithm of the apparent area was utilized. This was illustrated by literature data for the spleen.
thyroid and kidney. The theory behind such a log-log plot and its limitations, was outlined.

REFERENCES