APPROXIMATE EXPRESSION FOR THE GEOMETRIC RESPONSE AND THE INDEX OF RESOLUTION OF FOCUSED COLLIMATORS

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In a recent paper (*1*) Vetter showed that the image quality of a radionuclide scan perceived by an observer can be correlated with the half-flux resolution distance which is defined as the diameter of the circle centered at the ideal image point which circumscribed one half of the total flux contained in the image of a point source. In a second paper (*2*), he presents a practical method to evaluate the index of resolution of focused collimators at any depth. The method separates the imaging process of the collimator into two successive steps: one step involves the spread function resulting from one hole alone, and the second step involves the degradation of this spread function introduced by the multiplicity of holes. His results lack a rigorous theoretical foundation but are supported by experiments. They have been used in the evaluation of the tomographic camera (*3*). In this paper, it is shown that by means of the central limit theorem, Vetter's results can be derived from the rigorous expression for the geometric response of focused collimators. There is no need to introduce the half-flux resolution index. The results are then used to investigate the problem of scalloping.

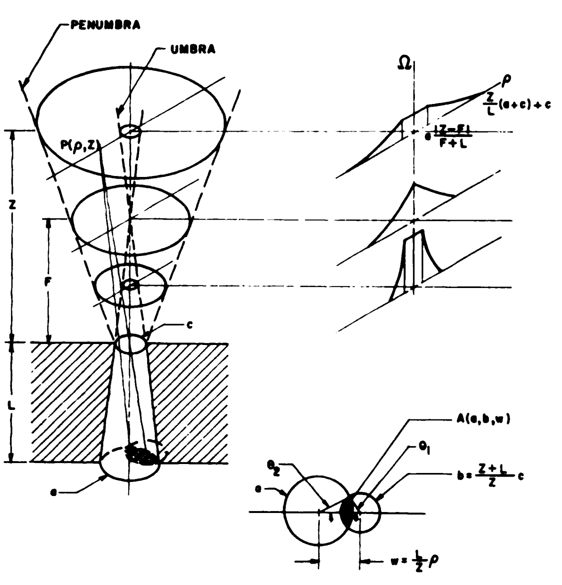

FIG. 1. Geometric response of tapered hole.

GEOMETRIC RESPONSE OF A SINGLE CHANNEL

Consider the tapered channel illustrated and described in Fig. 1. According to Brownell (*4*) its geometric response to a point source located at a distance Z from the entrance pupil of the channel and at a distance ρ from its axis is given by

$$\Omega(\rho,Z) = \frac{A(a,b,w)}{4\pi(L+Z)^2} \quad (1)$$

in which

$L =$ length of the channel
$A(a,b,w) =$ area of overlap of two circles of radii a and b, their center a distance w apart (see bottom of Fig. 1).
$a =$ radius of exit pupil
$c =$ radius of entrance pupil

$$b = c\,\frac{Z+L}{Z}$$

$$w = \frac{L}{Z}\,\rho.$$

The area of overlap of the two circles is the shadow of the entrance pupil on the exit pupil created by a point source at (ρ,Z). Let

$$P_r(\rho) = \begin{cases} 1 & \rho \leq r \\ 0 & \rho > r \end{cases} \quad (2)$$

be the zero-one function over the circle of radius r.

The area of overlap of two circles can be expressed in terms of a two-dimensional convolution integral written as

$$A(a,b,w) = P_a(w) ** P_b(w). \quad (3)$$

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In long hand the two-dimensional convolution integral of two circular symmetric functions takes the form

$$f(r) ** g(r) = \int_0^{\infty} \rho f(\rho) \int_{-\pi}^{+\pi} g(r^2 + \rho^2 - 2\rho r \cos\theta)^{1/2} d\theta \, d\rho. \quad (4)$$

For a fixed Z, the variation of Ω as a function of ρ is illustrated in the right of Fig. 1. It is zero outside the penumbra and constant inside the umbra region. In the penumbra its amplitude varies between zero and the constant value of the umbra region. The function Ω, however, is complex and difficult to use. A simple approximation is possible by means of the central limit theorem (*5*) which is used extensively in probability theory. In one dimension it states that a function f(x) which is the convolution of a large number of functions

$$f(x) = f_1(x) * f_2(x) * \ldots\ldots * f_n(x) \quad (5)$$

is approximately equal to a normal curve

$$f(x) = \frac{A(0)}{\sigma(2\pi)^{1/2}} e^{-(x-\eta)^2/2\sigma^2} \quad (6)$$

where

$$A(0) = \int_{-\infty}^{+\infty} f(x)dx = A_1(0) \times A_2(0) \times \ldots\ldots \times A_n(0)$$

$$A_i(0) = \int_{-\infty}^{+\infty} f_i(x)dx$$

is its area.

$$\eta = \eta_1 + \eta_2 \ldots\ldots \eta_n$$

$$\eta = \frac{\int_{-\infty}^{+\infty} x f(x)dx}{\int_{-\infty}^{+\infty} f(x)dx}$$

is its mean or first central moment.

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 \ldots\ldots + \sigma_n^2$$

$$\sigma^2 = \frac{\int_{-\infty}^{+\infty} (x-\eta)^2 f(x)dx}{\int_{-\infty}^{+\infty} f(x)dx}$$

is its variance or second central moment. The extension to two dimensions is readily accomplished. The central limit theorem yields a good approximation even for the convolution between two zero-one functions (*6*). For the zero-one function under consideration, it can be shown easily that

$$\eta = 0$$

$$\sigma = \frac{r^2}{2}.$$

Thus the geometric response can be approximated by

$$\Omega(\rho,Z) = \frac{\pi a^2 \pi b^2}{4\pi(L+Z)^2} \frac{1}{2\pi\left(\frac{a^2+b^2}{2}\right)} e^{-\{w^2/2[(a^2+b^2)/2]\}} = \quad (7)$$

$$\frac{\pi a^2 c^2}{4L^2} \frac{1}{2\pi\left(\frac{Z}{L}\right)^2\left(\frac{a^2+b^2}{2}\right)} e^{-\rho^2/2\{(Z/L)^2[(a^2+b^2)/2]\}}$$

For bell-shaped curves, it is common to define the index of resolution as the full width at half maximum or

$$d = 2.36\sigma. \quad (8)$$

For the tapered hole this leads to

$$d = 1.67 \frac{Z}{L} (a^2 + b^2)^{1/2} = 1.67 \frac{Z}{L}\left[a^2 + c^2\left(\frac{Z+L}{Z}\right)^2\right]^{1/2}. \quad (9)$$

The geometric efficiency due to a plane source of density s dps/cm², also known as the plane source sensitivity N, is obtained from

$$N = \int 2\pi\rho s \, \Omega(\rho,Z)d\rho. \quad (10)$$

Introducing the expression for Ω from Eq. 7 we obtain after some tedious but straightforward integration

$$N = \frac{\pi a^2 c^2}{4L^2} s, \quad (11)$$

a result often derived in a different manner (*7*).

DEGRADATION OF SPREAD FUNCTION DUE TO MULTIPLICITY OF HOLES

Consider a point source traversing uniformly the field of view of a multihole focusing collimator. Also the source moves in a plane such that the response from one channel does not overlap the response from adjacent channels. This corresponds to moving the source in plane A of Fig. 2. The resulting geometric response for one traverse is illustrated in the right of Fig. 2 under the heading "actual response." In rectangular coordinates the geometric response is a two-dimensional array of single-channel response curves or

$$H(x,y) = \Sigma\Sigma \, \Omega[(x - me_1 - ne_2)^2 + (y - mf_1 - nf_2)^2, Z]^{1/2}$$

$$= \Omega[(x^2 + y^2)^{1/2}, Z] ** s(x,y) \quad (12)$$

in which

$$s(x,y) = \Sigma\Sigma \, \delta(x - ne_1 - me_2) \, \delta(y - nf_1 - mf_2)$$

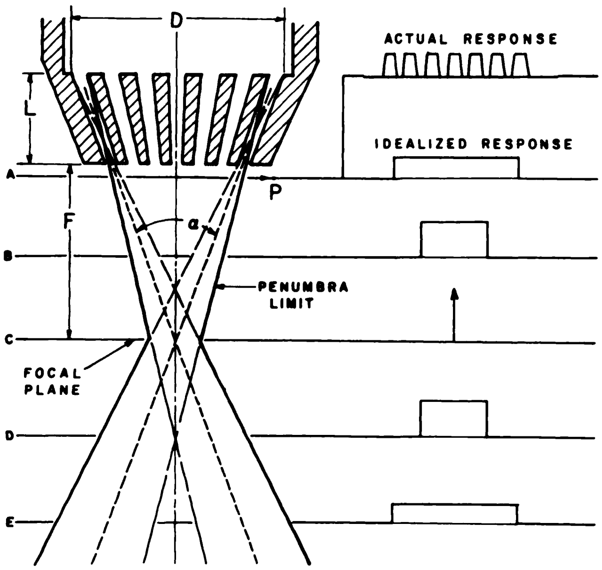

FIG. 2. Geometric response of multihole focused collimator.

describes the skew periodicity of the channels; thus m and n are integers, e_1, e_2, and f_1 and f_2 are spacing determined by the skew periodicity of the channels. The summation is over the number of channels. Since the above expression is a two-dimensional convolution integral, the central limit theorem can be used to obtain an approximation. Before doing so, we introduce a simplification for the function s(x,y). First we note that

$$\iint s(x,y)\,dx\,dy = M \qquad (13)$$

where M is the number of channels. Further, in the limit the number of channels becomes very large while their dimension becomes very small so that the field of view remains unchanged; we therefore have an idealized focused collimator and

$$s(x,y) = \frac{M}{\pi r_p^2} P_p(r) \qquad (14)$$

in which r_ρ is the radius of the field of view. This is the idealized spread function used by Vetter to account for the multiplicity of holes (*2*). Its mean and variance are

$$\eta_p = 0$$
$$\sigma_p^2 = \frac{1}{2} r_p^2. \qquad (15)$$

Notice that the factor of ½ is the result of the theory. It is the need for it that lead Vetter to introduce the half-flux concept. Examples of such spread functions at various distances from the face of the collimator are shown in the right of Fig. 2 under the heading "idealized response". With these simplifications, the overall geometric response at any depth is given by

$$H(\rho,Z) = \frac{M}{4\pi r_p^2 (L+Z)^2} \left[P_a\left(\frac{L}{Z}\rho\right) ** P_b\left(\frac{L}{Z}\rho\right)\right] ** P_p(\rho) \qquad (16)$$

which by virtue of the central limit theorem can be approximated by

$$H(\rho,Z) = \frac{\pi a^2 c^2 M}{4L^2} \frac{1}{2\pi \sigma_t^2} e^{-\rho^2/2\sigma_t^2}$$
$$\sigma_t^2 = \frac{1}{2}\left[\frac{Z^2}{L^2}(a^2+b^2) + r_p^2\right]. \qquad (17)$$

In view of these results, the resolution at any depth is given by

$$d = 1.67\left[\left(\frac{Z}{L}\right)^2 (a^2+b^2) + r_p^2\right]^{1/2}$$
$$= 1.67\left\{\left(\frac{Z}{L}\right)^2 \left[a^2 + c^2\left(\frac{Z+L}{Z}\right)^2\right] + r_p^2\right\}^{1/2} \qquad (18)$$

which is essentially the same as the expression obtained by Vetter. Vetter introduced the half-flux concept to derive his expression. In the derivation of Eq. 18 this concept is not necessary. As in the case of the single-channel collimator, the plane source sensitivity, N, is obtained from

$$N = \int 2\pi\rho s\, H(\rho,Z)\, d\rho = \frac{\pi a^2 c^2 M}{4L^2} \qquad (19)$$

in which s is the density of the plane sources in dps/cm². This is a well-known result. Equations 1 and 17 can also be used to estimate the depth of focus. For $\rho = 0$ we have

$$H(0,Z) = \frac{a^2 c^2 M}{4L^2} \frac{1}{\frac{Z^2}{L^2}(a^2+b^2) + r_p^2} \qquad (20)$$

in which r_p is the radius of the field of view at a distance Z — F from the focus or

$$r_p = |F - Z| \tan\frac{\alpha}{2}. \qquad (21)$$

D, α, and F are as indicated in Fig. 2. Equation 19 has the form of a Cauchy distribution; hence its second order moment is infinite. We may define the depth of focus as the full width at half maximum of Eq. 20. This leads to solving the equation

$$H(0,Z_{50}) = \frac{1}{2} H(0,F)$$

$$\frac{Z_{50}^2}{L^2} a^2 + \left(\frac{Z_{50}+L}{L}\right)^2 c^2 + (F - Z_{50})^2 \tan^2\frac{\alpha}{2} = 4\frac{F^2}{L^2} a^2 \qquad (22)$$

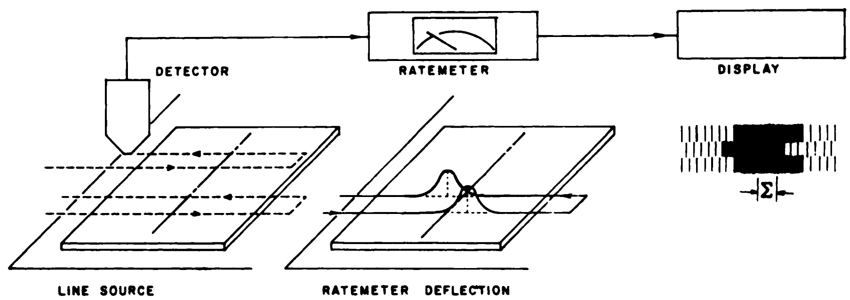

FIG. 3. Scalloping of line in linear motion scanning.

for Z_{50} which is the distance from the face of the collimator to the depth where the point source sensitivity is half maximum value. The interval between the two Z_{50} value is then the depth of focus.

SCALLOPING IN LINEAR MOTION SCANNING

We next use the central limit theorem to investigate the problem of scalloping. This problem has been investigated by means of modulation transfer function (*8,9*). Consider Fig. 3 in which the detector performs traverses over a line source in the $\pm$ x direction. Because of the collimator, the input signal to the ratemeter will be of the form

$$f(x) = e^{-x^2/2\sigma_c^2}. \tag{23}$$

Ratemeters are linear devices with spread function

$$h(x) = \frac{1}{(v\tau)} U(x)\, e^{-\frac{x}{(v\tau)}} \tag{24}$$

where

$$U(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

τ = is the time constant of the ratemeter

v = the scan velocity.

Because of the linearity, the output signal is given by

$$g(x) = f(x) * h(x) = \int f(\xi)\, h(x - \xi)\, d\xi. \tag{25}$$

This integral can be evaluated in close form. For simplicity we will use the central limit theorem. For the function in Eq. 24, we have

$$\eta = v\tau \qquad \sigma^2 = 2(v\tau)^2. \tag{26}$$

The output signal is also approximated by a Gaussian with mean and variance

$$\eta = v\tau \qquad \sigma^2 = \sigma_c^2 + 2(v\tau)^2. \tag{27}$$

The amount of scalloping Σ will be twice the displacement of the mean,

$$\Sigma = 2v\tau. \tag{28}$$

Keeping the amount of scalloping negligible is equivalent to

$$2(v\tau)^2 << \sigma_c^2 \qquad \tau << \frac{\sigma_c}{(2)^{1/2}v} \tag{29}$$

or the selection of the time constant is dictated by both the scan speed and the collimator's index of resolution.

CONCLUSIONS

A mathematical technique that simplifies the calculation of the collimator spread functions is presented. It is mainly used to establish a bridge between the rigorous theory and the simple method suggested by Vetter. The technique can be used to solve other problems. As an example, the problem of scalloping in linear motion scanning is solved. The technique can be used to solve many imaging problems without having to resort to modulation transfer functions.

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