## ON THE DIFFERENTIAL METHOD OF IMAGE FOCUSING

A recent article by Nagai, Fukuda and Iinuma (*1*) discussed a differential operator method of correcting for imperfect spatial resolution in digital radioisotope scan images formed by scanning systems with a "nearly . . . but not necessarily Gaussian" point-spread function. Although the first-order focusing correction applied by the authors is correct for system point-spread functions which are symmetric about the x and y axes under certain conditions shown below and is often a good approximation for nearly symmetric spread functions, the general deconvolution operator expressed by Eqs. 4–6 of the article in fact results in convergence to the object distribution if and only if the imaging system point-spread function is truly Gaussian. To my knowledge the general differential deconvolution technique has not been treated in the literature. The purpose of this note is to establish the proper form and conditions of applicability of a general differential deconvolution operator and to discuss the effects of image noise on the usefulness of the approach.

### THE GENERAL OPERATOR

The correct general differential deconvolution operator derived below is

$$1 + \left[ M_{10}\frac{d}{dx} + M_{01}\frac{d}{dy} \right] - \frac{1}{2!}\left[ (M_{20} - 2M_{10}^2)\frac{d^2}{dx^2} + 2\,(M_{11} - 2M_{10}M_{01})\frac{d^2}{dxdy} + (M_{02} - 2M_{01}^2)\frac{d^2}{dy^2} \right] + \left[ \begin{array}{l} \text{terms involving higher order moments} \\ \text{and derivatives} \end{array} \right]$$

in which the moments $M_{m,n}$ of the system point-spread function PSF(x,y) are defined by

$$M_{m,n} \equiv \iint_{-\infty}^{\infty} x^m y^n \, PSF(x,y)\, dxdy$$

and PSF(x,y) is assumed normalized to $M_{00} = \iint PSF(x,y)\, dxdy = 1$. Application of the differential deconvolution operator to an image distribution g(x,y) converges to the object distribution f(x,y) in the absence of noise if and only if the system frequency response function (i.e., the two-dimensional Fourier transform of the system point-spread function) is analytic and non-zero at all frequencies. Although higher order terms of the operator can be calculated by the method outlined below, no general form for the terms is apparent, and higher order terms quickly become quite complicated. I have calculated the third- and fourth-order terms, but limitations of space prevent their inclusion here.

The first few terms clearly show that if odd moments are non-zero—that is, if the point-spread function is asymmetric about its origin—then a first-order derivative term must be included, and that the general second-order coefficients are more complicated than those stated by Nagai *et al.* Higher order terms involve moments of order greater than 2, contrary to the general form given by Nagai *et al.*

### DERIVATION

Derivation of the general differential deconvolution operator will be outlined here for the one-dimensional case; extension to two-dimensional images is straightforward.

Since the Fourier transform of an nth order derivative of an image function g(x) is given by (*2*)

$$\mathcal{F}\left\{\frac{d^n g(x)}{dx^n}\right\} = (j\omega)^n\, G(\omega),$$

where $G(\omega)$ is the Fourier transform of g(x), expansion of an object function f(x) in terms of a series of derivatives of the image g(x) is clearly equivalent, in frequency space, to multiplying the image spectrum $G(\omega)$ by a complex polynomial in $\omega$. In the absence of noise, the object spectrum $F(\omega)$ is related to $G(\omega)$ by

$$F(\omega) = \frac{1}{M(\omega)}\, G(\omega)$$

where $M(\omega)$ is the system frequency response function (the Fourier transform of the system line-spread function).* Thus the derivative expansion should correspond, in frequency space, to a convergent polynomial expansion of $[M(\omega)]^{-1}$.

If and only if $M(\omega)$ is analytic and non-zero for all $\omega$, its reciprocal can be expanded in a unique

* For the two-dimensional case, the system frequency response function is the two-dimensional Fourier transform of the system point-spread function.

Taylor series about $\omega = 0$ which will converge for all $\omega$. Noting that $M(\omega)$ is the Fourier transform of the line-spread function LSF(x).

$$M(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x}\, \mathrm{LSF}(x)\,dx,$$

we have (*2*)

$$\frac{d^n M(0)}{d\omega^n} = (-j)^n \int_{-\infty}^{\infty} x^n\, \mathrm{LSF}(x)\,dx$$

$$\equiv (-j)^n \mu_n$$

where the $\mu_n$ are central moments of the system line-spread function. Taking the inverse Fourier transform of the product of the series and $G(\omega)$, one obtains the one-dimensional equivalent of the general operator above. The general *two*-dimensional operator involves moments of the system *point*-spread function.

The series expansion of the two-dimensional object distribution in terms of spatial derivatives of the image converges if and only if the Taylor expansion of $[M(\omega_x,\omega_y)]^{-1}$ in frequency space converges, and hence converges if and only if $M(\omega_x,\omega_y)$ is analytic and non-zero for all $\omega_x,\omega_y$.

One can show that if the system point-spread function is isotropic about its origin, then the two-dimensional differential deconvolution operator reduces to the much simpler form

$$1 - \frac{\mu_2}{2!}\nabla^2 + \frac{1}{4!}(6\mu_2^2 - \mu_4)\nabla^4$$
$$- \frac{1}{6!}(90\mu_2^3 - 30\mu_2\mu_4 + \mu_6)\nabla^6$$
$$+ \dots$$

in which the $\mu_m$ are even moments of the symmetric *line*-spread function LSF(x) defined by

$$\mu_m \equiv \int_{-\infty}^{\infty} x^m\, \mathrm{LSF}(x)\,dx$$

and

$$\nabla^{2n} \equiv \left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right]^n.$$

If the system point-spread and hence line-spread functions are truly Gaussian, then the operator reduces to the well-known inverse Laplacian (*3*). For this special case the "general" operator stated by Nagai *et al* is correct.

It is possible that in some situations a low-order correction using derivatives of the image might be useful even when the general method does not converge, or that low-order corrections weighted differently from the convergent terms might "improve" features of an image in some sense. The effect of such enhancement on overall frequency response can be predicted since the method can be viewed as multiplication of system response by a polynomial in frequency space. In any case, the technique must be applied with care.

### EFFECT OF IMAGE NOISE ON USEFULNESS OF THE METHOD

Since the differential deconvolution method is equivalent to multiplying the image spectrum by a polynomial in $\omega$, high-frequency image noise will obviously be enhanced very strongly. Although one can attempt to filter out high-frequency image noise by appropriate smoothing, excessive smoothing will tend to counteract the resolution-correcting effect of the deconvolution technique. The experimental scan image to which Nagai *et al* applied their first-order differential operator was composed of extraordinarily high count densities (up to 1,200 counts/mm²), yet noise enhancement in the processed image was still severe. The much poorer statistics of clinical scan images seem to leave the practical usefulness of the technique open to question.

### SUMMARY

A general differential deconvolution operator is derived and the conditions necessary for convergence in the absence of noise are established. The usefulness of the differential operator in the presence of image noise is discussed.

### ACKNOWLEDGMENT

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### REFERENCES

*1.* NAGAI, T., FUKUDA, N. AND IINUMA, T. A.: Computer-focusing using an appropriate Gaussian function. *J. Nucl. Med.* **10**:209, 1969.

*2.* PAPOULIS, A.: *The Fourier Integral and Its Applications*, McGraw-Hill, New York, 1962, p. 16.

*3.* KOVASZNAY, L. S. F. AND JOSEPH, H. M.: Image processing. *Proc. I.R.E.* **43**:560, 1955.

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## THE AUTHORS' REPLY

We wish to congratulate Dr. Metz on the excellent presentation of his Letter to the Editor on "Differential Operator Methods." We agree with his comments but would like to make a few remarks.