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Energy Absorption in Cylinders Containing an Axial Source[1,2]

J. C. Widman, M.A. and E. R. Powsner, M.D.[3]

Dearborn, Michigan

Calculation of energy absorption in tissue containing uniformly distributed isotopes requires knowledge of the emitted photon energy and the absorption fraction, ratio of energy absorbed to energy emitted. The energetics of most isotopes are known with greater precision than usually required. Although absorption coefficients are also known with more than enough precision (1, 2), calculation of the absorption fraction has long proven difficult because of the character of the integrations required for nonspherical volumes. For the sphere the problem is sufficiently tractable to have been the subject of several papers (3, 4, 5, 6, 7). In addition a limited number of other problems have been solved including: a few cubes, a series of two liter cylinders and one model patient, all for a uniformly distributed source in a medium with absorption coefficient $\mu = 0.028$ cm^{-1} (3); a table of values for cylinders of 11 lengths and eight radii and $\mu = 0.028$, containing a line source uniformly distributed along the cylinder axis (3); and various solutions for spheres and cylinders containing point sources (8, 9, 10, 11).

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[2]From the Radioisotope Service, Veterans Administration Hospital, Dearborn, Michigan.

[3]Chief, Radioisotope Service, Veterans Administration Hospital, Dearborn, Michigan, and Associate Professor of Pathology, Wayne State University School of Medicine, Detroit, Michigan.

Because of the availability of better computers, we have undertaken the calculation of the absorption fraction for cylinders with uniform axial and volume source distributions. This paper presents the solution for the cylinder with an axial source.

ANALYSIS

Consider a cylinder of radius R and height H (Fig. 1) with a source uniformly distributed along the axis with strength λ expressed as energy emission per unit length.

Let:

$U =$ photon energy absorbed

$\Phi =$ absorption fraction (ratio of absorbed energy to emitted energy)

$$= \frac{U}{\lambda H}$$

$\mu =$ linear absorption coefficient (exponential absorption assumed), and

$dU =$ energy absorbed in volume dV from energy emitted by source dz′

$=$ energy emitted by dz′ $\times$ fraction absorbed by dV

$$= \lambda dz' \, e^{-\mu r} \frac{dA}{4\pi r^2} \mu dr = \frac{\mu\lambda}{4\pi} \frac{e^{-\mu r}}{r^2} dV \, dz' .$$

Then,

$$U = \frac{\mu\lambda}{4\pi} \int dV \int \frac{e^{-\mu r}}{r^2} dz'.$$

To integrate the dV use cylindrical coordinates with origin on the axis at the base of the cylinder (see Fig. 1). To integrate the dz′, place the origin of the axial coordinate at point z:

$$U = \frac{\mu\lambda}{4\pi} \int_0^R \rho d\rho \int_0^{2\pi} d\phi \int_0^H dz \int_{-z}^{H-z} \frac{e^{-\mu(\rho^2 + z'^2)^{\frac{1}{2}}}}{\rho^2 + z'^2} dz'.$$

The value of the integral over ϕ is 2π. The absorption fraction

$$\Phi = \frac{U}{\lambda H}$$

$$= \frac{\mu}{2H} \int_0^R \rho d\rho \int_0^H dz \int_{-z}^{H-z} \frac{e^{-\mu(\rho^2 + z'^2)^{\frac{1}{2}}}}{\rho^2 + z'^2} dz' .$$

Because the integrand is an even function of z′ and does not depend on z

$$\Phi = \frac{\mu}{H} \int_0^R \rho d\rho \int_0^H dz \int_0^z \frac{e^{-\mu(\rho^2 + z'^2)^{\frac{1}{2}}}}{\rho^2 + z'^2} dz' \quad .$$

Substituting the dimensionless variables $E = \frac{H}{R}, k = \mu R, t = \frac{z'}{R}, s = \frac{\rho}{R}, y = \frac{z}{R}$

$$\Phi = \frac{k}{E} \int_0^E dy \int_0^y dt \int_0^1 \frac{e^{-k(s^2 + t^2)^{\frac{1}{2}}}}{s^2 + t^2} sds.$$

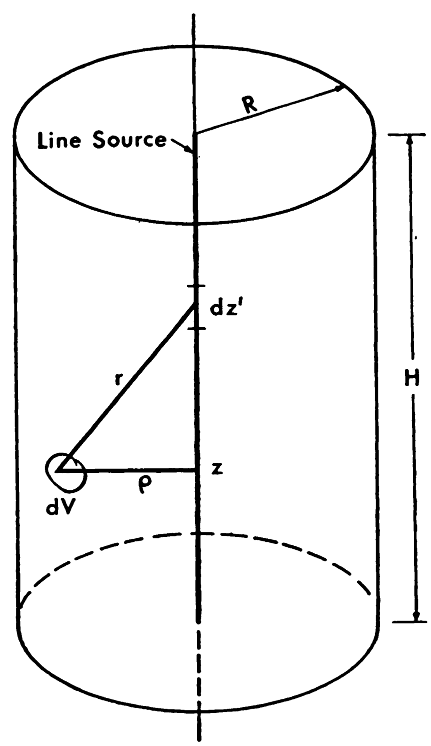

Fig. 1

To evaluate this, expand the exponential:

$$\Phi = \frac{k}{E}\int_o^E dy \int_o^y dt \int_o^1 \frac{sds}{s^2+t^2}\sum_{n=o}^{\infty}\frac{(-k)^n\,(s^2+t^2)^{\frac{n}{2}}}{n!}$$

$$= \frac{k}{E}\int_o^E dy \int_o^y dt \int_o^1 \frac{sds}{s^2+t^2} \qquad (1)$$

$$+ \frac{k}{E}\sum_{n=1}^{\infty}\frac{(-k)^n}{n!}\int_o^E dy \int_o^y dt \int_o^1 (s^2+t^2)^{\frac{n}{2}-1}\, sds\,. \qquad (2)$$

By reference to standard tables (*e.g.*, 12) it may be shown that

$$(1) = \frac{k}{2E}\left(2E\tan^{-1}E + E^2\log\frac{D}{E} - \log D\right)$$

where $D = (E^2+1)^{1/2}$. Proceeding with (2),

$$(2) = \frac{k}{E}\sum_{n=1}^{\infty}\frac{(-k)^n}{n\,.\,n!}\int_o^E dy \int_o^y dt \left[(t^2+1)^{\frac{n}{2}} - t^n\right]$$

$$= \frac{k}{E}\sum_{n=1}^{\infty}\frac{(-k)^n}{n\cdot n!}\int_o^E dy \int_o^y dt\,(t^2+1)^{\frac{n}{2}} \qquad (2a)$$

$$+ \sum_{n=1}^{\infty}\frac{(-kE)^{n+1}}{n\,(n+2)!}\,. \qquad (2b)$$

To evaluate (2a) use the formula

$$\int (t^2+1)^{\frac{n}{2}}dt = \frac{n!}{\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!}\sum_{j=0}^{\frac{n-1}{2}}\frac{(j!)^2\,t\,(t^2+1)^{j+\frac{1}{2}}}{(2j+1)!\,2^{n-2j}}$$

$$+ \frac{n!}{2^n\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!}\log\left(t+\sqrt{t^2+1}\right) \text{ for odd n, and}$$

$$= \frac{\left[\left(\frac{n}{2}\right)!\right]^2}{(n+1)!}\sum_{j=0}^{\frac{n}{2}}\frac{(2j)!\,2^{n-2j}}{(j!)^2}\,t\,(t^2+1)^j \qquad \text{for even n.}$$

Substituting and integrating over y

$$(2a) = \frac{k}{E}\sum_{n=2,4,6\ldots}^{\infty} \frac{k^n \left(\frac{n}{2}\right)!\left(\frac{n}{2}-1\right)!}{(n+1)!\,n!}\sum_{j=0}^{\frac{n}{2}} \frac{(2j)!\,2^{n-2j-2}}{(j+1)!\,j!}\left(D^{2j+2}-1\right)$$

$$-\frac{k}{E}\sum_{n=1,3,5\ldots}^{\infty} \frac{k^n}{n\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!}\sum_{j=0}^{\frac{n-1}{2}} \frac{(j+1)!\,j!}{(2j+3)!\,2^{n-2j-1}}\left(D^{2j+3}-1\right)$$

$$-\left[\log(D+E)-\frac{D-1}{E}\right]\sum_{n=1,3,5\ldots}^{\infty}\frac{k^{n+1}}{n2^n\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!}\ .$$

Combining (1), (2a) and (2b), and rewriting the summation indices over n,

$$\Phi = \frac{k}{2E}\left[2E\tan^{-1}E + E^2\log\frac{D}{E} - \log D\right] + \sum_{n=1}^{\infty}\frac{(-kE)^{n+1}}{n\,(n+2)!}$$

$$+\frac{k}{4E}\sum_{n=1}^{\infty}\frac{(2k)^{2n}\,n!\,(n-1)!}{(2n+1)!\,(2n)!}\sum_{j=0}^{n}\frac{(2j)!}{4^j\,(j+1)!\,j!}\left(D^{2j+2}-1\right)$$

$$-\frac{k^2}{E}\sum_{n=0}^{\infty}\left(\frac{k}{2}\right)^{2n}\frac{1}{(2n+1)\,(n+1)!\,n!}\sum_{j=0}^{n}\frac{4^j\,(j+1)!\,j!}{(2j+3)!}\left(D^{2j+3}-1\right)$$

$$-\frac{k^2}{2}\left[\log(D+E)-\frac{D-1}{E}\right]\sum_{n=0}^{\infty}\left(\frac{k}{2}\right)^{2n}\frac{1}{(2n+1)\,(n+1)!\,n!}\ . \qquad (3)$$

Values of the absorption fraction (eq. 3) are tabulated (Fig. 2, Table 1) for a wide range of the variables k and E.

As a byproduct of this work, the related problem of the absorption fraction for a cylinder with a point source at one end of the axis has been solved analytically. Using the same symbols as before,

$$\Phi = \frac{k}{2}\int_0^E dy \int_0^1 \frac{e^{-k\,(s^2+y^2)^{\frac{1}{2}}}}{s^2+y^2}\, s\,ds\ .$$

Integrating in a manner similar to that used above

$$\Phi = \frac{k}{2}\left[E\log\frac{D}{E}+\tan^{-1}E\right] + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-kE)^{n+1}}{n\,(n+1)!}$$

$$+\frac{kE}{4}\sum_{n=1}^{\infty}\frac{(2k)^{2n}\,n!\,(n-1)!}{(2n+1)!\,(2n)!}\sum_{j=0}^{n}\frac{(2j)!}{(j!)^2}\left(\frac{D}{2}\right)^{2j}$$

$$-DE\sum_{n=0}^{\infty}\left(\frac{k}{2}\right)^{2n+2}\frac{1}{(2n+1)\,(n+1)!\,n!}\sum_{j=0}^{n}\frac{(j!)^2}{(2j+1)!}(2D)^{2j}$$

$$-\log(D+E)\sum_{n=0}^{\infty}\left(\frac{K}{2}\right)^{2n+2}\frac{1}{(2n+1)\,(n+1)!\,n!}\ .$$

DISCUSSION

The absorption fractions presented here not only cover a wider range of cylinder sizes than have been previously published, but they also cover a range of values of the absorption coefficient. These data can easily be calculated by hand when k and kE are less than about three. For larger k or kE the calculation is lengthy and a digital computer is required. When a term was less than $3 \times 10,^{-3}$ the computation for a series was stopped. The estimated maximum error for the tabulated values of the absorption fraction is about 1×10^{-3}, a maximum which may be expected to occur infrequently. If required, additional or more accurate values may be computed without serious difficulty. The program written for an IBM 1620 can be made available if desired.

In dosimetric applications it is energy absorption which is important, in which case the appropriate μ to use is $\frac{\mu_{en}}{\rho}$ (13). The parameter μR is then $\left(\frac{\mu_{en}}{\rho}\right)\rho R$, where ρ here refers to the density of the absorbing medium.

The importance of these axial source data is somewhat greater than may be initially apparent. They provide an upper limit and estimate for the absorption fraction in the cylinder containing a source uniformly distributed throughout its volume. The estimate is particularly useful for the long, thin cylinder which may occur in limb or long-bone dose calculations, just the type of problem for which the spherical approximation is not useful. In addition the tabulated information can be used to obtain the energy absorbed by the cylindrical volume element along the axis of the cylinder when the source is uniformly distributed within the entire cylinder. What is commonly referred to as the reciprocity theorem (10, 14) states that the energy absorbed by one volume due to a radioactive

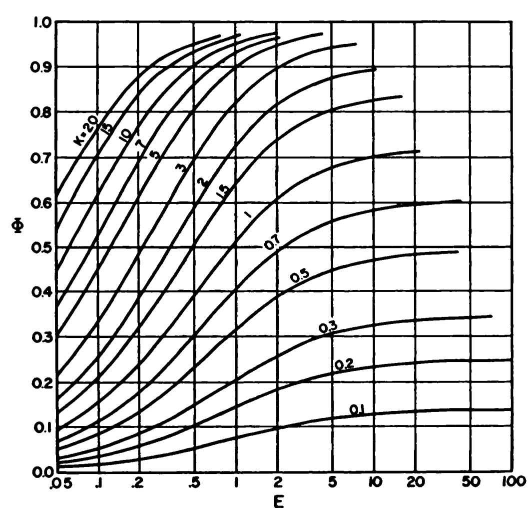

Fig. 2. Absorption fraction as a function of k = μR and E = H/R (μ = absorption coefficient, R = cylinder radius, H = cylinder height)

TABLE I

E/k	.05	.07	.10	.15	.20	.30	.40	.50	.70	1.0	1.5	2.0	3.0	4.0	5.0	7.0	10	15	20	30	40	50	100
0.02	.002	.003	.004	.005	.006	.008	.010	.011	.013	.016	.018	.020	.022	.024	.025	.026	.027	.028	.029	.029	.029	.030	.030
0.03	.003	.004	.006	.008	.009	.012	.014	.016	.020	.023	.027	.030	.033	.036	.037	.039	.040	.042	.042	.043	.044	.044	.044
0.04	.004	.006	.008	.010	.012	.016	.019	.022	.026	.031	.036	.040	.044	.047	.049	.051	.053	.055	.056	.057	.057	.058	.058
0.05	.006	.007	.009	.013	.015	.020	.024	.027	.032	.038	.045	.049	.055	.058	.061	.064	.066	.068	.069	.070	.071	.071	.072
0.07	.008	.010	.013	.018	.021	.028	.033	.038	.045	.053	.062	.068	.076	.081	.084	.088	.091	.093	.095	.096	.097	.097	.098
0.10	.011	.014	.019	.025	.030	.039	.047	.053	.064	.075	.088	.096	.106	.113	.117	.122	.126	.130	.132	.134	.135	.135	.136
0.15	.016	.021	.027	.037	.045	.058	.069	.078	.093	.110	.128	.140	.155	.163	.169	.176	.182	.186	.189	.191	.192	.193	.194
0.20	.022	.028	.036	.048	.059	.076	.091	.103	.122	.143	.166	.182	.200	.211	.218	.226	.233	.238	.241	.244	.245	.246	.248
0.30	.032	.041	.053	.071	.086	.111	.131	.148	.176	.205	.237	.258	.282	.296	.305	.316	.324	.331	.334	.337	.339	.339	
0.40	.041	.053	.069	.092	.111	.143	.169	.191	.225	.262	.301	.326	.355	.371	.381	.393	.402	.410	.413	.417	.418	.419	
0.50	.051	.065	.084	.112	.135	.174	.205	.230	.271	.314	.359	.387	.419	.437	.448	.461	.470	.478	.482	.485	.488		
0.70	.068	.088	.113	.149	.179	.229	.269	.301	.351	.404	.457	.490	.526	.546	.558	.571	.581	.590	.594	.598			
1.00	.093	.119	.152	.199	.238	.301	.350	.390	.451	.512	.528	.608	.646	.666	.678	.692	.702	.710	.714				
1.50	.129	.163	.208	.269	.319	.397	.457	.504	.573	.640	.703	.737	.774	.792	.803	.817	.825	.833					
2.00	.161	.202	.255	.326	.384	.472	.537	.587	.658	.724	.784	.816	.849	.865	.875	.886	.895						
3.00	.214	.267	.332	.416	.482	.579	.646	.695	.762	.821	.870	.895	.921	.934	.941	.950							
4.00	.259	.320	.393	.485	.554	.651	.716	.762	.821	.870	.910	.930	.950	.960	.966								
5.00	.299	.365	.443	.539	.609	.704	.764	.805	.857	.898	.931	.947	.963	.972									
7.00	.365	.439	.523	.622	.689	.774	.825	.859	.898	.928	.952	.964											
10.00	.443	.523	.610	.705	.765	.836	.876	.900	.929	.950	.966	.975											
15.00	.540	.621	.704	.788	.837	.889	.916	.933	.952	.967													
20.00	.610	.690	.765	.836	.876	.917	.937	.950	.964														

source uniformly distributed throughout a second volume would be equal to the energy absorbed in the second volume were the first to contain a source of the same uniform density. Under these circumstances it is also necessary that the absorption coefficients of the two volumes be equal. In our case, consider the axial line source within the cylinder to be a uniform volume distribution of density σ (energy emitted/unit volume) along a volume element of height H and area dA(dA < <A). The energy emitted is σHdA and the absorbed energy in the cylinder is $\Phi\sigma$HdA. Now consider the whole cylinder with the same source density so that the emitted energy is σHA, where $A = \pi R^2$. Using the reciprocity theorem the absorbed energy in the axial volume element HdA is $\Phi\sigma$HdA, and the absorption fraction for this axial volume element is

$$\Phi' = \Phi \frac{dA}{A} .$$

Using this formula the tabulated absorption fraction may be used to obtain the absorption fraction of the axial volume element.

SUMMARY

The absorbed photon energy is the total energy associated with the emitted photons times the absorption fraction. For a cylinder containing a radioactive isotope uniformly distributed along its axis, the absorption fraction is derived analytically assuming only that the linear absorption coefficient is constant throughout the cylinder. Values of the absorption fraction are tabulated as a function of the linear absorption coefficient and cylinder dimensions.

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